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RADIATION INSTABILITY OF A CIRCULAR CYLINDER IN A UNIFORM FLOW OF A TWO-LAYER FLUID†

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A solution of the plane linear problem of the oscillations of a horizontal circular cylinder in a uniform flow of a two-layer unbounded fluid is obtained using the method of multipole expansions. The direction of the flow is perpendicular to the cylinder axis. The whole cylinder lies in the upper or lower layer. The fluid is assumed to be ideal and incompressible, the flow in each layer being a potential one. All the components of the radiation load (the apparent masses and damping coefficients) are determined and the regions of existence of radiation instability are found, depending on the flow velocity for a cylinder suspended by horizontal and vertical elastic links. By solving the integro-differential equation numerically the relative oscillations of the body under specified initial conditions are found.

In a study of the hydrodynamical characteristics in the linear theory of a moving submerged body it has been found that the damping coefficients can be negative. This means that, as in the case of flutter in aeroelasticity, the motion of a rigid body with oscillatory degrees of freedom in a fluid can exhibit radiation instability, that is, amplification of oscillations can occur at the expense of the energy of translational motion.

Negative damping coefficients have been investigated [1] when computing the hydrodynamic loads acting on an ellipsoid moving uniformly under the free surface of a homogeneous fluid and simultaneously undergoing oscillations in one of the six degrees of freedom. It was noted that this only occurs at very high velocities of the body and indicates that the energy of translational motion (or flow) can be converted both into wave emission and oscillations of their source. When a submerged body moves near a sharp density jump in a stratified fluid, for example, a two-layer fluid with a small density difference between the layers, negative damping coefficients occur even for relatively moderate motions of the body [2].

The loss of stability of the relative position of equilibrium of an oscillator moving at constant velocity and interacting with gravitation waves has been investigated in [3–6]. An instability effect was discovered by computing the energy losses for a moving medium [3] and a circular cylinder [4] under the interface boundary in an unbounded two-layer fluid, and also for a sphere under the free surface of a homogeneous fluid of finite depth or an unbounded uniformly stratified fluid [5]. An asymptotic analysis has been carried out [6] for the integro-differential equation describing the oscillations of a circular cylinder under the free surface of an unbounded homogeneous fluid. The dipole approximation of a moving body has been used [3–6] and the possibility of oscillations of a body with only one horizontal degree of freedom has been considered. It is well known that the dipole approximation in the radiation problem gives a singular solution at the resonance frequency, and only taking into account a body of non-zero volume can lead to a finite solution. This conclusion was obtained in [7] for a semibounded homogeneous fluid with a free surface and can be extended to the case of a two-layer unbounded fluid.

The approach presented below is based on the complete solution of the linear radiation problem for a circular cylinder in the flow of a two-layer fluid by the method of multipole expansions (this solution is similar to that obtained earlier in [8, 9]). The possibility of non-decaying oscillations of a cylinder with two degrees of freedom is investigated using the results of the hydrodynamic theory of pitching [10] using the methods of the theory of aeroelasticity [11].

1. FORMULATION OF THE PROBLEM OF THE OSCILLATIONS OF A BODY IN A UNIFORM FLOW OF A TWO-LAYER FLUID

In the unperturbed state the upper fluid layer of density ρ_1 occupies the domain $|x| < \infty, y < 0$ and the lower fluid layer of density $\rho_2 = \rho_1(1 + \varepsilon)$ $(\varepsilon > 0)$ occupies the domain $|x| < \infty, y > 0$, where

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x is the horizontal coordinate and y is the vertical coordinate. There is a uniform flow around the body with velocity U in the negative direction of the x axis. The cylinder undergoes small oscillations in the two possible degrees of freedom with frequency ω.

Assuming that the perturbed oscillatory motion of the fluid is steady, we can seek the total velocity potential of the whole wave motion in the form

$$\Phi^{(s)}(x,y,t) = -Ux + U\overline{\Phi}^{(s)}(x,y) + \mathrm{Re}\sum_{j=1}^{2}\eta_j\Phi_j^{(s)}(x,y)e^{i\omega t}$$

where $\overline{\Phi}^{(s)}$ are the velocity potentials corresponding to the uniform motion of the body with unit velocity, $\Phi_j^{(s)}(j = 1, 2)$ characterise the radiation potentials due to the forced pitch of the body in the horizontal direction ($j = 1$) and vertical direction ($j = 2$), η_j are the upper oscillation amplitudes of the body, the superscript $s = 1, 2$ is introduced for the upper and lower layer, respectively, and t is the time.

For time-independent potentials inside the fluid

$$\Delta\overline{\Phi}^{(1)} = 0 \ \ (y>0), \ \ \Delta\overline{\Phi}^{(2)} = 0 \ \ (y<0) \tag{1.1}$$

According to linear wave theory, the boundary conditions on the interface have the form

$$\partial^2[(1+\varepsilon)\overline{\Phi}^{(2)} - \overline{\Phi}^{(1)}]/\partial x^2 + \varepsilon\mu\partial\overline{\Phi}^{(1)}/\partial y = 0, \ \ \partial\overline{\Phi}^{(1)}/\partial y = \partial\overline{\Phi}^{(2)}/\partial y \ \ (y=0) \tag{1.2}$$

where $\mu = g/U^2$, and g is the acceleration due to gravity. In the far field we must impose the radiation condition, which means that no upstream waves are present, and also the condition for damping of the wave process as $|\,y\,| \to \infty$

$$\partial\overline{\Phi}^{(s)}/\partial x \to 0 \ \ (x\to\infty), \ \ \left|\partial\overline{\Phi}^{(s)}/\partial x\right| < \infty \ \ (x\to-\infty) \ \ (s=1,2) \tag{1.3}$$

$$\partial\overline{\Phi}^{(1)}\partial y \to 0 \ \ (y\to\infty), \ \ \partial\overline{\Phi}^{(2)}/\partial y \to 0 \ \ (y\to-\infty) \tag{1.4}$$

On the circular contour $S : x^2 + [y + (-1)^q h]^2 = a^2$ we pose the impermeability condition

$$\partial\overline{\Phi}^{(q)}/\partial n = n_1 \ \ (x, y \in S) \tag{1.5}$$

where $\mathbf{n} = (n_1, n_2)$ is the inner normal to the contour S, a is the cylinder radius, and h is the distance between the cylinder axis and the interface boundary ($h > a$), $q = 1$ ($q = 2$) when the cylinder is placed in the upper (lower) layer.

The components of the radiation potentials satisfy the equations

$$\Delta\Phi_j^{(1)} = 0 \ \ (y>0), \ \ \Delta\Phi_j^{(2)} = 0 \ \ (y<0) \ \ (j=1,2) \tag{1.6}$$

similar to (1.1), and the boundary conditions

$$N[(1+\varepsilon)\Phi_j^{(2)} - \Phi_j^{(1)}] + \varepsilon g\partial\Phi_j^{(1)}/\partial y = 0, \ \ \partial\Phi_j^{(1)}/\partial y = \partial\Phi_j^{(2)}/\partial y \ \ (y=0) \tag{1.7}$$

$$\partial\Phi_j^{(1)}/\partial y \to 0 \ \ (y\to\infty), \ \ \partial\Phi_j^{(2)}/\partial y \to 0 \ \ (y\to-\infty) \tag{1.8}$$

$$\partial\Phi_j^{(q)}/\partial n = i\omega n_j - Um_j \ \ (x,y\in S) \tag{1.9}$$

where

$$N \equiv (U\partial/\partial x - i\omega)^2, \ \ (m_1, m_2) = \partial(\partial\overline{\Phi}^{(q)}/\partial x, \ \partial\overline{\Phi}^{(q)}/\partial y)/\partial n$$

The radiation conditions for $\Phi_j^{(s)}$ require that a wave can propagate upstream only when its phase velocity is positive and the group velocity is greater than the velocity of the body. Otherwise wave motion can only be present behind the body. The hydrodynamic forces $\mathbf{F} = (F_1, F_2)$ acting on the cylinder can be determined by integrating the fluid motion (without the hydrostatic term) $p = -\rho_q(\partial\Phi^{(q)}/\partial t + |\nabla\Phi^{(q)}|^2/2)$ around S. It is convenient to replace this integral by the sum

$$F_j = F_{sj} + \mathrm{Re}(F_{rj}e^{i\omega t}) \ \ (j = 1, 2)$$

where F_{sj} are the time-independent forces (the wave resistance and lift) acting on a stationary body in a uniform flow, and F_{rj} are the radiation forces, which are usually written in matrix form (for more details, see for example [2, 10])

$$F_{rj} = \eta_1 T_{j1} + \eta_2 T_{j2}$$

The quantities T_{jk} represent the complex force acting in the j direction and due to the sinusoidal oscillations of the body with the unit amplitude in direction k. They can be represented in the form $T_{jk} = \omega^2 A_{jk} - i\omega B_{jk}$. The real quantities A_{jk} and B_{jk} are known as the apparent mass and damping coefficients, respectively.

Introducing the polar system of coordinates r, θ with origin at the centre of the contour S

$$x = r\sin\theta,\quad y + (-1)^q h = r\cos\theta \quad (q = 1, 2)$$

and taking into account that

$$n_1 = -\sin\theta,\quad n_2 = -\cos\theta \tag{1.10}$$

for a circular contour, we obtain [9]

$$\left\|\begin{matrix} F_{s1} \\ F_{s2} \end{matrix}\right\| = \frac{\rho_q U^2}{2a}\int_0^{2\pi}\left[\frac{\partial(\overline{\Phi}^{(q)} - x)}{\partial\theta}\right]^2 \left\|\begin{matrix} \sin\theta \\ \cos\theta \end{matrix}\right\| d\theta \tag{1.11}$$

$$T_{jk} = \rho_q a \int_0^{2\pi} \frac{\partial\Phi_j^{(q)*}}{\partial n}\Phi_k^{(q)} d\theta \tag{1.12}$$

where the asterisk denotes the complex conjugate.

2. SOLUTION OF THE STEADY-STATE PROBLEM

The solution of the steady-state problem by the method of multipole expansions was obtained in [8]. We shall present only the basic results of this solution, which are necessary to study the radiation problem (Section 3).

When considering a cylinder in the lower layer we represent the solution of Eqs (1.1) with boundary conditions (1.2)–(1.4) in the form

$$\overline{\Phi}^{(1)} = (1+\gamma)\,\mathrm{Re}\sum_n \frac{a^n C_n}{(n-1)!}\left[\mathrm{p.v.}\int_0^\infty \frac{k^n}{k-\nu} e^{-k(y+h+ix)}dk + i\pi\nu^n e^{-\nu(y+h+ix)}\right]$$

$$\overline{\Phi}^{(2)} = \mathrm{Re}\sum_n a^n C_n \left\{\frac{e^{-in\theta}}{r^n} - \frac{\gamma}{(n-1)!}\left[\mathrm{p.v.}\int_0^\infty k^{n-1}\frac{k+\mu}{k-\nu}e^{k(y-h-ix)}dk + \right.\right.$$
$$\left.\left. + i\pi\nu^{n-1}(\nu+\mu)e^{\nu(y-h-ix)}\right]\right\};\quad \nu = \mu\gamma,\quad \gamma = \varepsilon/(2+\varepsilon)$$

(p.v. denotes the principal value of the integral; summation over n and later on over m is from 1 to ∞).

To take boundary condition (1.5) into account we use the well-known relation

$$\exp[k(y+h\pm ix)] = 1 + \sum_m \frac{(kr)^m}{m!}\exp(\pm im\theta) \tag{2.1}$$

The value of the potential in the lower layer can be written in the form

$$\overline{\Phi}^{(2)} = \mathrm{Re}\left\{\sum_m\left[\frac{a^m}{r^m}C_m - \gamma\sum_n \frac{a^n r^m}{m!(n-1)!}I^+_{n+m-1}C_n\right]e^{-im\theta} - \right.$$
$$\left. -\gamma\sum_n \frac{a^n}{(n-1)!}C_n I^+_{n-1}\right\},\quad I^\pm_m = \mathrm{p.v.}\int_0^\infty k^m \frac{k\pm\mu}{k-\nu}e^{-2kh}dk \pm i\pi\nu^m(\nu\pm\mu)e^{-2\nu h} \tag{2.2}$$

We can compute $I_m^{\pm}$ from the recursive formula

$$I_m^{\pm} = \left(\frac{m}{2h} \pm \mu\right)\frac{(m-1)!}{(2h)^m} + \nu I_{m-1}, \quad I_0^{\pm} = \frac{1}{2h} \pm (\nu \pm \mu)e^{-2\nu h}[i\pi \mp \mathrm{Ei}(2\nu h)] \tag{2.3}$$

Ei is the integral exponential function of real argument [12].

Differentiating (2.2) with respect to r and using (1.5) and (1.10), we obtain the system of linear equations for C_n

$$C_m + \gamma \sum_n \frac{a^{m+n} C_n}{m!(n-1)!} I_{n+m-1}^{+} = -i\delta_{m1} \tag{2.4}$$

where δ_{m1} is the Kronecker delta. Substituting (2.4) into (2.2) and differentiating with respect to θ, we obtain

$$\partial(\overline{\Phi}^{(2)} - x)/\partial\theta = -2\,\mathrm{Re}\sum_m imC_m e^{-im\theta} \tag{2.5}$$

for $r = a$.

When considering a cylinder in the upper layer, as previously we obtain

$$\overline{\Phi}^{(1)} = \mathrm{Re}\left\{\sum_m\left[\frac{a^m}{r^m}C_m + (-1)^m\gamma\sum_n\frac{(-a)^n r^m}{m!(n-1)!}I_{n+m-1}^{-}C_n\right]e^{-im\theta} + \right.$$
$$\left. + \gamma\sum_n\frac{(-a)^n}{(n-1)!}C_n I_{n-1}^{-}\right\} \tag{2.6}$$

The system from which to determine C_n differs from (2.4) in that γ is replaced by $-\gamma$, a by $-a$, and I_{n+m+1}^{+} by I_{n+m+1}^{-}. An expression similar to (2.5) holds in this case also. It follows that in both cases the time-independent forces in (1.11) have the form

$$F_{s2} + iF_{s1} = \frac{2\pi\rho_q U^2}{a}\sum_m m(m+1)C_m^{*}C_{m+1}$$

3. SOLUTION OF THE RADIATION PROBLEM

We shall seek the potential in the lower layer for the radiation problem (1.6)–(1.8) using a radiation condition of the form (the cylinder is in the lower layer)

$$\Phi_j^{(2)} = \sum_n a^n\left[D_{jn}^{-}\left(\frac{e^{-in\theta}}{r^n} - F_{2n}^{-}\right) + D_{jn}^{+}\left(\frac{e^{in\theta}}{r^n} - F_{2n}^{+}\right)\right] \quad (j = 1, 2) \tag{3.1}$$

$$F_{2n}^{\pm} = \frac{1}{(n-1)!}\left\{\mathrm{p.v.}\int_0^\infty k^{n-1}\left[\gamma + \frac{1+\gamma}{\sqrt{1\pm 4\tau}}\left(\frac{k_1^{\pm}}{k-k_1^{\pm}} - \frac{k_2^{\pm}}{k-k_2^{\pm}}\right)\right]\times\right.$$
$$\left.\times e^{k(y-h\pm ix)}dk \mp \frac{i\pi(1+\gamma)}{\sqrt{1\pm 4\tau}}[(k_1^{\pm})^n e^{k_1^{\pm}(y-h\pm ix)} \mp (k_2^{\pm})^n e^{k_2^{\pm}(y-h\pm ix)}]\right\}$$

$$k_1^{\pm} = \frac{\nu}{2}(1 \pm 2\tau + \sqrt{1\pm 4\tau}), \quad k_2^{\pm} = \frac{\nu}{2}(1 \pm 2\tau - \sqrt{1\pm 4\tau}), \quad \tau = \frac{\omega U}{\gamma g}$$

The real quantities k_1^{-}, k_2^{-} exist only when $\tau \leqslant 1/4$. Otherwise the last term in braces for F_{2n}^{-} vanishes.

To compute the second derivatives in boundary conditions (1.9) we use the following relations, as in [9]

$$\int_0^{2\pi}\frac{\partial^2\overline{\Phi}^{(2)}}{\partial n\partial x}e^{im\theta}d\theta = -\frac{im}{a^2}\int_0^{2\pi}\frac{\partial(\overline{\Phi}^{(2)} - x)}{\partial\theta}e^{im\theta}\sin\theta d\theta = \frac{i\pi m}{a^2}P_m^{-} \tag{3.2}$$

$$\int_0^{2\pi} \frac{\partial^2 \overline{\Phi}^{(2)}}{\partial n \partial y} e^{im\theta} d\theta = -\frac{im}{a^2} \int_0^{2\pi} \frac{\partial(\overline{\Phi}^{(2)} - x)}{\partial \theta} e^{im\theta} \cos\theta d\theta = -\frac{\pi m}{a^2} P_m^+$$

$$P_m^{\pm} = (m+1)C_{m+1} \pm (m-1)C_{m-1}$$

Substituting (2.1) into (3.1), differentiating with respect to r, and using (1.9), taking (3.2) into account, we obtain the following systems of linear equations from which to determine the unknown coefficients $D_{jn}^{\pm}$

$$D_{jm}^{\pm} + \sum_n \frac{a^{m+n}}{m!(n-1)!} H_{n+m-1}^{\pm} D_{jn}^{\pm} = X_{jm}^{\pm} \quad (j = 1,2) \tag{3.3}$$

where

$$X_{1m}^{-} = (a\omega\delta_{m1} - iUP_m^- / a)/2, \quad X_{1m}^{+} = (iUP_m^{-*} / a - a\omega\delta_{m1})/2$$

$$X_{2m}^{-} = (UP_m^+ / a - ia\omega\delta_{m1})/2, \quad X_{2m}^{+} = (UP_m^{+*} / a - ia\omega\delta_{m1})/2 \tag{3.4}$$

$$H_m^{\pm} = \text{p.v.}\int_0^{\infty} k^m \left[\gamma + \frac{1+\gamma}{\sqrt{1 \pm 4\tau}}\left(\frac{k_1^{\pm}}{k - k_1^{\pm}} - \frac{k_2^{\pm}}{k - k_2^{\pm}}\right)\right] e^{-2kh} dk +$$

$$+ \frac{i\pi(1+\gamma)}{\sqrt{1 \pm 4\tau}}[(k_2^{\pm})^{m+1} e^{-2k_2^{\pm}h} \mp (k_1^{\pm})^{m+1} e^{-2k_1^{\pm}h}]$$

To evaluate the integrals in (3.4) we use recursive formulae similar to (2.3). When $\tau > 1/4$, the last term in H_m^- should be omitted, the integral is to be understood in the ordinary sense, and the integral exponential function of the complex argument E_1 should be used in the recursive formula. In this case

$$H_m^- = \frac{\gamma m!}{(2h)^{m+1}} + \frac{2(1+\gamma)}{\sqrt{4\tau - 1}} \operatorname{Im}[k_1^- V_m(k_1^-)]$$

where

$$V_m(k_1^-) = \int_0^{\infty} \frac{k^m}{k - k_1^-} e^{-2kh} dk = W_m(-2hk_1^-)/(2h)^m$$

$$W_m(z) = (m-1)! - zW_{m-1}(z) \quad (m \geqslant 2), \quad W_1(z) = 1 - ze^z E_1(z)$$

On the body surface for $r = a$

$$\Phi_j^{(2)} = \sum_m [(2D_{jm}^- - X_{jm}^-)e^{-im\theta} + (2D_{jm}^+ - X_{jm}^+)e^{im\theta}] + c_j \quad (j = 1,2) \tag{3.5}$$

where $c_{1,2}$ are constants similar to the last term in (2.6).

If the cylinder is placed in the upper layer, then, by analogy with (3.1)

$$\Phi_j^{(1)} = \sum_n a^n \left[D_{jn}^- \left(\frac{e^{-in\theta}}{r^n} + F_{1n}^-\right) + D_{jn}^+ \left(\frac{e^{in\theta}}{r^n} + F_{1n}^+\right)\right] \quad (j = 1,2)$$

where

$$F_{1n}^{\pm} = \frac{(-1)^n}{(n-1)!} \{\text{p.v.}\int_0^{\infty} k^{n-1} \left[\gamma - \frac{1-\gamma}{\sqrt{1 \mp 4\tau}}\left(\frac{k_1^{\mp}}{k - k_1^{\mp}} - \frac{k_2^{\mp}}{k - k_2^{\mp}}\right)\right] \times$$

$$\left. \times e^{k(\mp ix - y - h)} dk - \frac{i\pi(1-\gamma)}{\sqrt{1 \mp 4\tau}}[(k_2^{\mp})^n e^{k_2^{\mp}(\mp ix - y - h)} \pm (k_1^{\mp})^n e^{k_1^{\mp}(\mp ix - y - h)}]\right\}$$

Using (1.9), we obtain a system of equations which differs from (3.3) in that a is replaced by $-a$, $H_{n+m-1}^{\pm}$ by $H_{n+m-1}^{\mp}$, and γ by $-\gamma$ in the expression for $H_m^{\pm}$. The expressions for $\Phi_j^{(1)}$ on the body surface are the same as in (3.5).

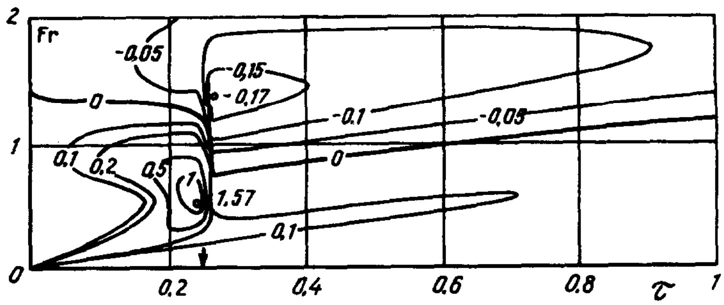

Fig. 1.

After solving system (3.3) we determine the radiation load coefficients T_{jk}. Substituting (1.9) and (3.5) into (1.12), we obtain

$$T_{jk} = 2\pi\rho_q \sum_m m[X_{jm}^{-*}(2D_{km}^{-} - X_{km}^{-}) + X_{jm}^{+*}(2D_{km}^{+} - X_{km}^{+})] \tag{3.6}$$

The diagonal damping coefficients B_{jj} can also be expressed in terms of the potential characteristic $\Phi_j^{(q)}$ on the interface in the far field by means of the law of conservation of energy. According to the relations for a two-layer fluid [2] and a homogeneous fluid (see for example [9]), B_{jj} can, in general, be represented as the sum of four terms of constant sign corresponding to the waves which occur in the far field, the contributions of all waves apart from k_1^+ being positive. Physically, this means that the wave k_1^+ ensures an energy flux towards the body, while the wave energy flux for the three elastic waves is directed away from the body. It is possible for negative values of b_{jj} to appear only when the contribution of k_1^+ predominates.

Tables with the values of all the components of the hydrodynamic load for a homogeneous fluid are given in [9]. To obtain values with an accuracy up to five significant figures it suffices to use only eight terms in (3.3).

The isolines of the damping coefficient $\overline{B}_{11} = \omega B_{11}/(\rho_2 \gamma g a)$ as a function of the Froude number $\mathrm{Fr} = U/\sqrt{(\gamma g a)}$ and τ are presented in Fig. 1 for a two-layer fluid for $\varepsilon = 0.03$, and $h/a = 2$ (the cylinder is in the lower medium). The positions and values of the extrema of the given function are indicated by dots and numbers, the value $\tau = 1/4$ being shown by an arrow.

In Table 1 we give the maximum values ($\overline{B}_{jj}^{+}$) and minimum values ($\overline{B}_{jj}^{-}$) of $\overline{B}_{jj} = \omega B_{jj}/(\rho_q \gamma g a)$, their location ($\tau_+$, Fr_+ and τ_-, Fr_-, respectively), and also the values of Fr_* at which the first negative values are attained by the diagonal damping coefficients for the following positions of the cylinder when $h/a = 2$: A—under the free surface of a homogeneous fluid ($\varepsilon \to \infty$) and B—in the lower or upper layer in a two-layer fluid ($\varepsilon = 0.03$). It is clear that the behaviour of $\overline{B}_{jj}$ is quite similar in all these cases in the chosen dimensionless variables. The values given in the table are obtained for $0 \leqslant \tau \leqslant 1$, $0 \leqslant \mathrm{Fr} \leqslant 2$ on a grid with steps $\Delta\tau = 0.02$, $\Delta\mathrm{Fr} = 0.05$. In [4] the value $\mathrm{Fr}_* = 0.74$ was given for $\overline{B}_{11}$ in a two-layer fluid using the dipole approximation.

Table 1

	A		B	
	$j = 1$	2	1	2
τ_+	0.22	0.20	0.24	0.24
Fr_+	0.55	0.65	0.50	0.55
$\overline{B}_{jj}^{+}$	1.97	2.53	1.57	1.81
τ_-	0.26	0.26	0.26	0.26
Fr_-	1.30	1.15	1.35	1.25
$\overline{B}_{jj}^{-}$	–0.35	–0.32	–0.17	–0.17
Fr_*	0.75	0.70	0.75	0.75

4. THE MOTION OF A CYLINDER ATTACHED TO ELASTIC LINKS

Suppose that the cylinder is attached to linear elastic springs with four stationary points arranged pairwise on the vertical and horizontal lines passing through its centre. In the unperturbed state the cylinder is at rest, the time-independent load (the wave resistance and lift) being balanced by the action of the springs. We introduce a fixed system of coordinates x_1, x_2 passing through the centre of the unperturbed cylinder and parallel to the original system x, y. Suppose that the cylinder is displaced by a small distance with respect to the given position of equilibrium at the initial time $t = 0$, so the coordinates of its centre are equal to $\mathbf{x}(0) = (x_1^0, x_1^0)$ ($|\mathbf{x}(0)| \ll a$). The subsequent motion $\mathbf{x}(t) = (x_1(t), x_2(t))$ of the cylinder is assumed to be oscillatory about the fixed mean position. With all the forces approximated linearly, it can be described by the system of integro-differential equations (see for example [13])

$$\sum_{k=1}^{2}(M_{jk}+a_{jk})\ddot{x}_k+b_{jk}\dot{x}_k+\int_0^t Q_{jk}(s)\dot{x}_k(t-s)ds+C_{jk}x_k=0 \quad (j=1,2)$$

$$a_{jk}=\lim_{\omega\to\infty}A_{jk}(\omega),\ \ b_{jk}=\lim_{\omega\to\infty}B_{jk}(\omega),\ \ Q_{jk}(t)=\frac{2}{\pi}\int_0^\infty[B_{jk}(\omega)-b_{jk}]\cos\omega t d\omega \tag{4.1}$$

with initial conditions

$$x_1(0)=x_1^0,\quad x_2(0)=x_2^0,\quad \dot{x}_1(0)=\dot{x}_2(0)=0$$

The dot denotes differentiation with respect to t, M_{jk} and C_{jk} are the diagonal coefficients of the masses and restoring forces with $M_{11} = M_{22} = M$, $C_{11} = \lambda_1$, $C_{22} = \lambda_2$, $M = \pi\rho_0 a^2$ is the linear mass of the cylinder, ρ_0 is the density of the material of the cylinder, λ_1 and λ_2 are the stiffnesses of the horizontal and vertical springs, respectively, and the kernel $Q_{jk}(t)$ is the delay function.

In the limiting case as $\omega \to \infty$ the behaviour of the apparent mass and damping coefficients can be obtained by analysing (3.6). For the problem under consideration $a_{11} = a_{22} = a_0$, where a_0 is the apparent mass of the circular cylinder in a weightless stationary two-layer fluid, and $a_{12} = a_{21} = {}_0$. The off-diagonal damping coefficients depend on the flow velocity and $b_{12} = -b_{21}$, $b_{11} = b_{22} = 0$.

When the cylinder undergoes steady-state harmonic oscillations with frequency ω, its motion satisfies the following equations, analogous to a mechanical oscillator (see for example [10])

$$\sum_{k=1}^{2}[M_{jk}+A_{jk}(\omega)]\ddot{x}_k+B_{jk}(\omega)\dot{x}_k+C_{jk}x_k=0 \quad (j=1,2)$$

Substituting into these equations the relations

$$x_k(t)=\mathrm{Re}[\eta_k(\omega)e^{i\omega t}] \quad (k=1,2) \tag{4.2}$$

for the trajectories of the body, we obtain the relations

$$\sum_{k=1}^{2}[(M_{jk}+A_{jk})\omega^2-i\omega B_{jk}-C_{jk}]\eta_k=0 \quad (j=1,2) \tag{4.3}$$

from which to determine the complex amplitudes η_k.

The simplest task is to study the behaviour of a cylinder with only one degree of freedom. Suppose that there are no vertical springs and the cylinder oscillates only along the horizontal x_1 axis, remaining at a fixed distance h from the interface. In this case (4.3) can be reduced to a single equation

$$(M+A_{11})\omega^2-i\omega B_{11}-\lambda_1=0 \tag{4.4}$$

which implies that steady-state oscillations are only possible at a frequency ω_* when $B_{11}(\omega_*) = 0$ and the stiffness of the horizontal spring is equal to

$$\lambda_1=\omega_*^2[M+A_{11}(\omega_*)] \tag{4.5}$$

Following the harmonicity hypothesis, which is widely used in the theory of aeroelasticity, we replace the real quantity ω in the coefficients of (4.4) by a complex number $\omega + i\xi$ putting $|\xi| \ll 1$. The

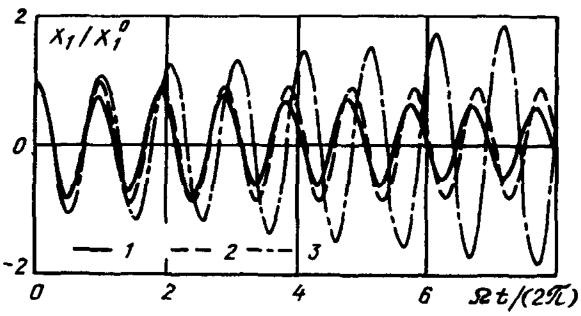

Fig. 2.

hydrodynamic loads are functions of the real argument, as before. By (4.2) the oscillation are damped when $\xi > 0$ and amplified when $\xi < 0$. Separating the resulting equation into the real and imaginary parts, we obtain $2\xi = \omega B_{11}/(M + A_{11})$. Consequently, fixing the stiffness of the horizontal springs and varying the free-stream velocity in the unsteady problem on the development of the oscillations of the body following an initial displacement, we obtained damped oscillations for sufficiently small Froude numbers, a steady-state oscillation regime with frequency ω_* when condition (4.5) is satisfied, and increasing oscillations as the free-stream velocity is increased further.

The results of the numerical solution of Eqs (4.1) by the method of finite-differences [14] in the case of one degree of freedom are presented in Fig. 2 for a two-layer fluid (the parameters correspond to Fig.1) for various Froude numbers Fr = 1/2; 1; 2 (curves 1–3) and a spring with stiffness $\Lambda_1 = \lambda_1/(\rho_2\gamma a) = 3$, $\rho_0 = \rho_2$, $\Omega = \sqrt{(\lambda_1/(M + a_{11}))}$, $a_{11}/(\pi\rho_2 a^2) = 0.9982$. When Fr = 1/2 the values of B_{11} are positive and the oscillations decay. The value Fr = 1 is closest to the null isoline of the damping coefficient in Fig. 1. The case Fr = 2 corresponds to negative values of B_{11} and the oscillation amplitudes increase.

The case of two degrees of freedom can also be analysed using the harmonicity hypothesis. In place of ω we substitute $\omega + i\xi$ in (4.3) and we equate the determinant of this homogeneous linear system to zero. Separation of the resulting complex equation into the real and imaginary parts leads to a system of linear equations from which to determine ω and ξ,

$$\begin{aligned}&(\omega^4 - 6\omega^2\xi^2 + \xi^4)d_1 - \xi(3\omega^2 - \xi^2)d_2 - (\omega^2 - \xi^2)d_3 - \xi d_4 + \lambda_1\lambda_2 = 0\\&4\xi(\omega^2 - \xi^2)d_1 + (\omega^2 - 3\xi^2)d_2 - 2\xi d_3 + d_4 = 0\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}&d_1 = f_1 f_2 - A_{21}A_{12}, \quad d_2 = A_{21}B_{12} + A_{12}B_{12} - f_2 B_{11} - f_1 B_{22}\\&d_3 = \lambda_1 f_2 + \lambda_2 f_1 + B_{11}B_{22} - B_{12}B_{21}, \quad d_4 = \lambda_1 B_{22} + \lambda_2 B_{11}\\&f_j = M + A_{jj} \quad (j = 1, 2)\end{aligned}$$

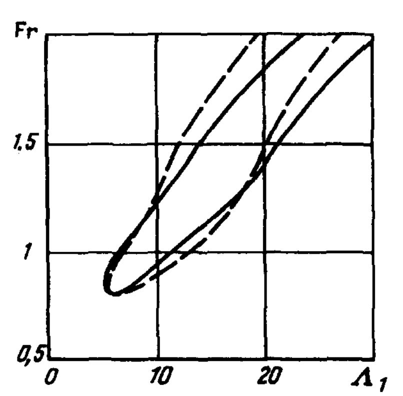

Fig. 3.

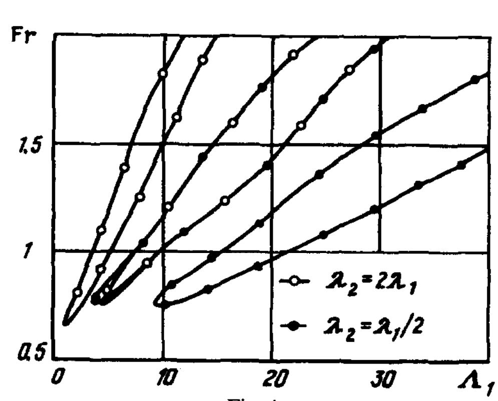

Fig. 4.

The second equation in (4.6) is a polynomial of degree three in ξ. It is not difficult to find its real roots for any $\omega > 0$. After substituting these values of ξ into the first equation (4.6), we obtain an equation from which to determine ω. This can be solved numerically when Fr, λ_1, λ_2 are fixed. The regimes in which $\xi < 0$ corresponds to the values of ω found in this way are regarded as unstable.

The results of this numerical analysis are presented in Fig. 3 for $\lambda_1 = \lambda_2$. The solid curve is constructed for the parameters corresponding to Fig. 2, and the dashed curve for a one-layer fluid with free surface ($\varepsilon \to \infty$, $h/a = 2$). The region inside the solid curve corresponds to the values of Fr and Λ_1 for which radiation instability occurs for a two-layer fluid. Similarly, the region inside the dashed curve corresponds to a homogeneous fluid.

The results for a two-layer fluid in the case when the stiffness of the horizontal and vertical springs are different are presented in Fig. 4 for $\lambda_2 = 2\lambda_1$ and $\lambda = \lambda_1/2$. In each of these two cases there are two ranges of values of Fr and Λ_1 for which radiation instability occurs.

Despite the fact that the linear approximation used cannot describe the behaviour of a cylinder undergoing oscillations with arbitrary amplitude, it may prove useful when determining the parameters of the given oscillatory system which are safest from the viewpoint of radiational instability.

The proposed method can be extended to the case of a fluid of finite depth. It can also be used to study the radiational instability of a medium for which the hydrodynamic loads can be determined using point-like multipoles in the same way as in the earlier approach presented in [15].

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